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Schwinger Method for 3-Dimensional Time Dependent Quadratic Systems

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Abstract We present an operatorial technique that uses Schwinger action principle, by means of quantum canonical transformations. This technique enables us to decouple the Hamiltonian of 3-dimensional time dependent quadratic systems (3D TDQS) with linear terms and to transform it to that of free particles, in order to determine exactly the relative propagator of the system. The study is made in this paper for general 3D systems with 3 hybrid anisotropic and varying coupling terms (containing both of position and momentum components), for 3D systems with one varying hybrid coupling term, and for 3D systems with no coupling terms. Two physically relevant examples are presented to illustrate the concrete application of the general formula obtained in this study, those of charged particles in scalar and vector potentials.

Keywords Calculus of variations · Quantum mechanics · Formalism

1 Introduction

Time-Dependent Quadratic Systems (TDQS) are widely dealt with in physics such for: particles in electromagnetic field, waves in nonlinear media, interaction of atoms or molecules with electromagnetic radiation. And this is the origin of the growing interest in propagators of such systems. Functional methods are usually employed to derive propagators for TDQS via path integrals [1–6]. In the other hand, Schwinger action principle, which is an operatorial method [7, 8] has not taken interest as it should have, and it was not confronted to all realistic and existing problems, nevertheless, it gives the opportunity to find propagators for more complicated problems with more exactness and rigorousness than with functional

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methods. However, propagators for TDQS were determined exactly for some general cases with Schwinger action principle using semi classical method [9, 10] or analytical method via resolution of Heisenberg differential equations [11–13].

Canonical transformations are fruitful tools to simplify the study of complicated systems. They succeeded to transform the Hamiltonian of general 1D TDQS to that of a free particle following Schwinger action principle [14], and they were able to reduce, for example, the motion of Bloch electrons in homogeneous magnetic fields to at most two dimensions in the general three dimensional case for any arbitrary rational fields [15], and many other studies were made in the same way [6, 16–18].

For N-dimensional systems, coupling terms should be eliminated with canonical transformations in order to transform Hamiltonians with coupling terms to those of simpler uncoupled systems [16]; we can refer here to the case of the elimination of electron–phonon interaction in order to obtain the effective electron–electron scattering [19]. Therefore, diagonalization of Hamiltonians by means of time-dependent canonical transformations is of great utility to make this decoupling, and it was applied for dissipative systems in quantum optics and in condensed matter systems [20].

The aim of this paper is to establish a generalisation of pure operatorial Schwinger action principle method by means of canonical transformations, to determine exact propagators for 3D TDQS. And as Heisenberg differential equations restrict the generalisation of Schwinger action principle, the introduction of quantum canonical transformations avoid the resolution of these equations, and simplify the Hamiltonian by making a diagonalization in matrices of its quadratic terms.

In [14], we have realized successfully the generalization of Schwinger action principle via canonical transformations for the most general 1D TDQS. We go on in our generalization, and we formulate this generalization for 3D TDQS with hybrid varying couplings. We will preserve here the same notation as for [14]. However, for a 3D system, with position vector and momentum vector expressed respectively as

$$\hat{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{p}} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix},$$

Schwinger action principle is expressed in terms of the variation of the action operator $\hat{W}$ as

$$\delta\langle \mathbf{x}'', t''|\mathbf{x}', t'\rangle = \frac{i}{\hbar}\langle \mathbf{x}'', t''|\delta\hat{W}|\mathbf{x}', t'\rangle$$

with

$$\delta\hat{W} = \hat{\mathbf{p}}''\delta\hat{\mathbf{x}}'' - \hat{\mathbf{p}}'\delta\hat{\mathbf{x}}' - \hat{H}''\delta t' + \hat{H}'\delta t', \tag{1}$$

where $\hat{\mathbf{x}}'$ and $\hat{\mathbf{x}}''$ are respectively initial and final position vector operators at initial and final instants t' and t''. The action operator should be rearranged by means of the commutator related to $\hat{\mathbf{x}}''$ and $\hat{\mathbf{x}}'$, to obtain the well ordered action such as $\hat{\mathbf{x}}''$ stands in the left of $\hat{\mathbf{x}}'$. This step enables us to get the eigen value of $\delta\hat{W}$, and to make the integration. The propagator of the system is then obtained by means of Schwinger action principle, using (1), as

$$\langle \mathbf{x}'', t''|\mathbf{x}', t'\rangle = C\exp\left(\frac{i}{\hbar}w(\mathbf{x}'', \mathbf{x}', t'', t')\right), \tag{2}$$

where w is the eigen value of the well ordered action operator. The coefficient C is determined from the normalization condition:

$$\lim_{t''\to t'}\langle x_i'', t''|x_i', t'\rangle = \delta(x_i'' - x_i').$$

In this work we are going to use quantum canonical transformations: $(\hat{\mathbf{x}}, \hat{\mathbf{p}}) \to (\hat{\mathbf{X}}, \hat{\mathbf{P}})$, in evaluating propagators in the same way as in the previous work [14]. And it will be necessary to determine the quantum generating functions $\hat{F}_1$, and $\hat{F}_2$, in order to specify the appropriate canonical transformations, so that for a certain direction $\hat{x}_i$:

$$\hat{p}_i = \frac{\delta \hat{F}_2(\hat{\mathbf{x}}, \hat{\mathbf{P}}, t)}{\delta \hat{x}_i}, \qquad \hat{X}_i = \frac{\delta \hat{F}_2(\hat{\mathbf{x}}, \hat{\mathbf{P}}, t)}{\delta \hat{P}_i} \tag{3}$$

and

$$\hat{F}_1(\hat{\mathbf{x}}, \hat{\mathbf{X}}, t) = -\frac{1}{2}(\hat{\mathbf{X}}\hat{\mathbf{P}} + \hat{\mathbf{P}}\hat{\mathbf{X}}) + \hat{F}_2(\hat{\mathbf{x}}, \hat{\mathbf{P}}, t). \tag{4}$$

The function playing the role of the Hamiltonian in the new coordinates set is then expressed in terms of the generating function as

$$\hat{\hat{H}} = \hat{H} + \frac{\delta \hat{F}_2}{\delta t}. \tag{5}$$

In this case, the variation of the action operator is expressed in the new coordinate set as:

$$\delta \hat{W}(\hat{\mathbf{P}}, \hat{\mathbf{X}}, t) = \hat{\mathbf{P}}''\delta\hat{\mathbf{X}}'' - \hat{\mathbf{P}}'\delta\hat{\mathbf{X}}' - \hat{\hat{H}}''\delta t'' + \hat{\hat{H}}'\delta t' + \delta(\hat{F}_1'' - \hat{F}_1'). \tag{6}$$

Schwinger action principle is then applied to find the propagator in the new set, which is subsequently turned to the former set, and written in terms of the well ordered action obtained from (6) in the new set of coordinates:

$$\langle \mathbf{x}'', t''|\mathbf{x}', t'\rangle = C\exp\left(\frac{i}{\hbar}w(\mathbf{X}'', \mathbf{X}', t'', t')\right), \tag{7}$$

A here is determined from the normalization condition [14].

This paper is organized as follows. In Sect. 2 we construct a model in which we determine exactly the propagator for general 3D TDQS with linear terms by means of Schwinger action principle, after making a diagonalization in matrices of quadratic terms in the Hamiltonian via canonical transformations. In Sects. 3, 4, and 5, we will make applications of the general result obtained in Sect. 2, in treating successively the cases of: a general system with one varying hybrid coupling, then a general system with three varying anisotropic hybrid couplings, and finally a general system with three linear independent movements. Section 6 is devoted to the summary.

2 Propagators for 3D TDQS with Linear Terms

The Hamiltonian of a general 3D TDQS with linear terms is expressed as:

$$\hat{H} = \hat{\mathbf{p}}\mathbf{G}_1\hat{\mathbf{p}} + \hat{\mathbf{x}}\mathbf{G}_2\hat{\mathbf{x}} + \hat{\mathbf{x}}\mathbf{G}_3\hat{\mathbf{p}} + \hat{\mathbf{p}}\mathbf{G}_4\hat{\mathbf{x}} + \mathbf{z}_1\mathbf{x} + \mathbf{z}_2\hat{\mathbf{p}} + \delta(t), \tag{8}$$

where $\delta(t)$ is an arbitrary time dependent function; $\mathbf{z}_1$, $\mathbf{z}_2$ and are time dependent vectors; $\mathbf{G}_1$ and $\mathbf{G}_2$ are diagonal matrices; and $\mathbf{G}_3$ and $\mathbf{G}_4$ are antisymmetric matrices so that

$$
\begin{aligned}
&\mathbf{G}_1 = \begin{pmatrix} g_1(t) & 0 & 0 \\ 0 & g_2(t) & 0 \\ 0 & 0 & g_3(t) \end{pmatrix}, \qquad \mathbf{G}_2 = \begin{pmatrix} h_1(t) & 0 & 0 \\ 0 & h_2(t) & 0 \\ 0 & 0 & h_3(t) \end{pmatrix}, \qquad \mathbf{G}_3 = \mathbf{G}_4^T, \\
&\mathbf{G}_3 = \mathbf{G}_{31} + \mathbf{G}_{32}, \qquad \mathbf{G}_{31} = \begin{pmatrix} k_1(t) & 0 & 0 \\ 0 & k_2(t) & 0 \\ 0 & 0 & k_3(t) \end{pmatrix}, \\
&\mathbf{G}_{32} = \frac{1}{2}\begin{pmatrix} 0 & l_3(t) & -l_2(t) \\ -l_3(t) & 0 & l_1(t) \\ l_2(t) & -l_1(t) & 0 \end{pmatrix}.
\end{aligned}
\tag{9}
$$

In the generalization made for 1D TDQS which has a linear motion [14], we have just introduced a translation in the canonical transformations. For the 3D system described above, the hybrid coupling term is due to angular momentum, for this reason we establish the canonical transformations from a translation and a rotation. Consequently, we consider the following canonical transformations:

$$
\begin{cases} \hat{\mathbf{x}} = (2m)^{1/2}\mathbf{DRT}\hat{\mathbf{X}} + \mathbf{f}_1, \\ \hat{\mathbf{p}} = (2m)^{-1/2}\mathbf{D}^{-1}\mathbf{RT}^{-1}\hat{\mathbf{P}} + \mathbf{f}_2, \end{cases} \tag{10}
$$

where m is the mass of the particle; $\mathbf{R}$ a rotation matrix which is an antisymmetric one so that its transposed matrix $\mathbf{R}^T = \mathbf{R}^{-1}$, and it is used to delete the hybrid coupling terms; $\mathbf{f}_1$ and $\mathbf{f}_2$ are time dependent vectors used to delete the linear terms; $\mathbf{D}$ and $\mathbf{T}$ are time dependent diagonal matrices, where

$$
\mathbf{T} = \begin{pmatrix} \rho_1(t) & 0 & 0 \\ 0 & \rho_2(t) & 0 \\ 0 & 0 & \rho_2(t) \end{pmatrix} \quad \text{and} \quad \mathbf{D}^2 = \mathbf{G}_1. \tag{11}
$$

Equations (3) and (4) enable us to find the following generating functions:

$$
\begin{cases} \hat{F}_2 = (8m)^{-1/2}(\hat{\mathbf{x}}\mathbf{D}^{-1}\mathbf{RT}^{-1}\hat{\mathbf{P}} + \mathbf{D}^{-1}\mathbf{RT}^{-1}\hat{\mathbf{P}}\hat{\mathbf{x}}) + \mathbf{f}_2\hat{\mathbf{x}} - (2m)^{-1/2}\hat{\mathbf{P}}\mathbf{T}^{-1}\mathbf{R}^{-1}\mathbf{D}^{-1}\mathbf{f}_1 \\ \quad = \dfrac{1}{2}(\hat{\mathbf{X}}\hat{\mathbf{P}} + \hat{\mathbf{P}}\hat{\mathbf{X}}) + (2m)^{1/2}\mathbf{f}_2\mathbf{DRT}\hat{\mathbf{X}} + \mathbf{f}_1\mathbf{f}_2, \\ \hat{F}_1 = (2m)^{1/2}\mathbf{f}_2\mathbf{DRT}\hat{\mathbf{X}} + \mathbf{f}_1\mathbf{f}_2. \end{cases} \tag{12}
$$

Hence the Hamiltonian is transformed, after using (5) with the properties of the transposed matrix and those of rotation and diagonal matrices, to

$$
\begin{aligned}
\hat{H}(\hat{\mathbf{P}}, \hat{\mathbf{X}}, t) &= (2m)^{-1}\hat{\mathbf{P}}\mathbf{D}_1\hat{\mathbf{P}} + 2m\hat{\mathbf{X}}\mathbf{D}_2\hat{\mathbf{X}} + \hat{\mathbf{X}}\mathbf{D}_3\hat{\mathbf{P}} + \hat{\mathbf{P}}\mathbf{D}_4\hat{\mathbf{X}} + (2m)^{1/2}\hat{\mathbf{X}}\mathbf{TR}^{-1}\mathbf{DK}_1 \\
&\quad + (2m)^{-1/2}\hat{\mathbf{P}}\mathbf{T}^{-1}\mathbf{R}^{-1}\mathbf{D}^{-1}\mathbf{K}_2 + \Gamma(t),
\end{aligned} \tag{13}
$$

where

$$
\mathbf{D}_1 = \mathbf{T}^{-2}, \tag{14}
$$

$$
\mathbf{D}_2 = \mathbf{TR}^{-1}\mathbf{DG}_2\mathbf{DRT}, \tag{15}
$$

$$\mathbf{D}_3 = \mathbf{T}\mathbf{R}^{-1}\mathbf{D}\left(\frac{1}{2}\dot{\mathbf{D}}^{-1} + \mathbf{G}_3\mathbf{D}^{-1}\right)\mathbf{R}\mathbf{T}^{-1} + \frac{1}{2}(\mathbf{R}^{-1}\dot{\mathbf{R}}\mathbf{T}^{-1} + \dot{\mathbf{T}}^{-1}), \tag{16}$$

$$\mathbf{D}_4 = \mathbf{D}_3^T = \mathbf{T}^{-1}\mathbf{R}^{-1}\left(\frac{1}{2}\dot{\mathbf{D}}^{-1} + \mathbf{D}^{-1}\mathbf{G}_4\right)\mathbf{D}\mathbf{R}\mathbf{T} + \frac{1}{2}(\mathbf{T}^{-1}\dot{\mathbf{R}}^{-1}\mathbf{R} + \dot{\mathbf{T}}^{-1})\mathbf{T}, \tag{17}$$

$$\mathbf{K}_1 = \dot{\mathbf{f}}_2 + 2\mathbf{G}_3\mathbf{f}_2 + 2\mathbf{G}_2\mathbf{f}_1 + \mathbf{z}_1, \tag{18}$$

$$\mathbf{K}_2 = -\dot{\mathbf{f}}_1 + 2\mathbf{G}_4\mathbf{f}_1 + 2\mathbf{G}_1\mathbf{f}_2 + \mathbf{z}_2, \tag{19}$$

and

$$\Gamma(t) = \dot{\mathbf{f}}_2\mathbf{f}_1 + \mathbf{f}_1\mathbf{G}_2\mathbf{f}_1 + \mathbf{f}_2\mathbf{G}_1\mathbf{f}_2 + 2\mathbf{f}_1\mathbf{G}_3\mathbf{f}_2 + \mathbf{z}_1\mathbf{f}_1 + \mathbf{z}_2\mathbf{f}_2 + \delta(t). \tag{20}$$

We define here: $\dot{\mathbf{f}}_i = \frac{\partial \mathbf{f}_i}{\partial t}$, $\dot{\mathbf{D}} = \frac{\partial \mathbf{D}}{\partial t}$, $\dot{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial t}$, $\dot{\mathbf{T}} = \frac{\partial \mathbf{T}}{\partial t}$. Now we will solve the following differential equations system in order to drop the linear terms in the new Hamiltonian by nullifying their coefficients such as

$$\begin{cases} \dot{\mathbf{f}}_1 - 2\mathbf{G}_4\mathbf{f}_1 - 2\mathbf{G}_1\mathbf{f}_2 = \mathbf{z}_2, \\ \dot{\mathbf{f}}_2 + 2\mathbf{G}_3\mathbf{f}_2 + 2\mathbf{G}_2\mathbf{f}_1 = -\mathbf{z}_1. \end{cases} \tag{21}$$

The solutions of these equations depend on the values of the matrices elements. And to delete the coupling terms, i.e. the elements of the matrix $\mathbf{G}_{32}$, we nullify, in the matrix $\mathbf{D}_3$, the elements D_{3ij} generated by the matrix $\mathbf{G}_{32}$, i.e. for $i \neq j$, so that we leave the diagonal elements D_{3ii}. To be done we consider

$$\mathbf{A} = \mathbf{D}_3 = \mathbf{D}_3^T = \mathbf{D}_4 = \frac{1}{2}\mathbf{T}(\dot{\mathbf{T}}^{-1} + \mathbf{R}^{-1}(\mathbf{D}\dot{\mathbf{D}}^{-1} + 2\mathbf{G}_{31})\mathbf{R}\mathbf{T}^{-1}). \tag{22}$$

This step nullifies the existing coupling in the system dynamic, but the matrix $\mathbf{R}$, which is a non diagonal matrix, generates, in the general case, new dynamic or static dynamical or static coupling terms what complicates the problem. For this reason, and as this technique is developed, for the moment, only in the case of hybrid couplings, the choice of the system is restricted to those systems containing a hybrid coupling term and a partial isotropy in the quadratic terms for the same directions. This means that systems with three varying couplings require 3-dimensional quadratic isotropy in order to be a solvable problem. Otherwise, a system with anisotropic quadratic terms in three dimensions, i.e. a system with three independent linear movements, does not permit the existence of any hybrid coupling term. This condition of isotropy is not imposed to the linear terms. In verifying this condition, the matrix $\mathbf{A}$ would be a diagonal one. We obtain then the following differential equation:

$$\dot{\mathbf{R}} = -2\mathbf{D}\mathbf{G}_{32}\mathbf{D}^{-1}\mathbf{R}. \tag{23}$$

The system will be decoupled once this differential equation is resolved. Consequently, we write the simplified Hamiltonian and the action operator variation in the new coordinate set as

$$\begin{cases} \hat{H}(\hat{\mathbf{P}}, \hat{\mathbf{X}}, t) = (2m)^{-1}\hat{\mathbf{P}}\mathbf{D}_1\hat{\mathbf{P}} + 2m\hat{\mathbf{X}}\mathbf{D}_2\hat{\mathbf{X}} + \hat{\mathbf{X}}\mathbf{A}\hat{\mathbf{P}} + \hat{\mathbf{P}}\mathbf{A}\hat{\mathbf{X}} + \Gamma(t), \\ \delta\hat{W} = \hat{\mathbf{P}}''\delta\hat{\mathbf{X}}'' - \hat{H}''\delta t'' - \hat{\mathbf{P}}'\delta\hat{\mathbf{X}}' + \hat{H}'\delta t' + \delta[\hat{F}_1'' - \hat{F}_1']. \end{cases} \tag{24}$$

We introduce now the following canonical transformation:

$$\begin{cases} \hat{\hat{\mathbf{X}}} = \hat{\mathbf{X}}, \\ \hat{\hat{\mathbf{P}}} = \hat{\mathbf{P}} + 2m\mathbf{D}_1^{-1}\mathbf{A}\hat{\mathbf{X}}. \end{cases} \tag{25}$$

It has been said previously that $\mathbf{T}$ and $\mathbf{A}$ are diagonal matrices, leading to $\mathbf{D}_1^{-1}\mathbf{A} = (\mathbf{D}_1^{-1}\mathbf{A})^T$. Consequently, we determine from (3) and (4) the following generating functions for this transformation:

$$\begin{cases} \hat{\hat{F}}_2 = \frac{1}{2}(\hat{\mathbf{X}}\hat{\hat{\mathbf{P}}} + \hat{\hat{\mathbf{P}}}\hat{\mathbf{X}}) - m\hat{\mathbf{X}}\mathbf{D}_1^{-1}\mathbf{A}\hat{\mathbf{X}}, \\ \hat{\hat{F}}_1 = -m\hat{\mathbf{X}}\mathbf{D}_1^{-1}\mathbf{A}\hat{\mathbf{X}} \end{cases} \tag{26}$$

Then (5) enables us to transform the Hamiltonian in (24) to

$$\hat{\hat{H}}(\hat{\hat{\mathbf{P}}}, \hat{\mathbf{X}}, t) = \frac{1}{2m}\hat{\hat{\mathbf{P}}}\mathbf{D}_1\hat{\hat{\mathbf{P}}} + \frac{m}{2}\hat{\mathbf{X}}\mathbf{D}_1\mathbf{\Delta}\hat{\mathbf{X}} + \Gamma(t), \tag{27}$$

where

$$\mathbf{\Delta} = 4\mathbf{D}_1^{-1}\left(\mathbf{D}_2 - \mathbf{A}\mathbf{D}_1^{-1}\mathbf{A} - \frac{1}{2}(\dot{\mathbf{D}}_1^{-1}\mathbf{A} + \mathbf{D}_1^{-1}\dot{\mathbf{A}})\right). \tag{28}$$

When substituting (14–16) in (28) and taking into consideration the fact that matrices $\mathbf{D}$, $\mathbf{G}_2$, $\mathbf{G}_{31}$, and $\mathbf{T}$, are diagonal ones, with the condition of isotropy that should be verified totally or partially in certain circumstances as explained above. Thus, the matrix $\mathbf{R}$ will have no effect on the matrix $\mathbf{A}$ in (22). We obtain then

$$\mathbf{\Delta} = \mathbf{T}^3(\ddot{\mathbf{T}} + \mathbf{T}(4\mathbf{D}^2\mathbf{G}_2 - 4\mathbf{G}_{31}^2 + 4\mathbf{D}^{-1}\dot{\mathbf{D}}\mathbf{G}_{31} - 2\mathbf{D}^{-2}\dot{\mathbf{D}}^2 + \mathbf{D}^{-1}\ddot{\mathbf{D}} - 2\dot{\mathbf{G}}_{31})) \tag{29}$$

which is also a diagonal matrix. Reaching this stage we have successfully decoupled the Hamiltonian into 3 independent linear systems, what enables us to write the new Hamiltonian as:

$$\hat{\hat{H}}(\hat{\hat{\mathbf{P}}}, \hat{\mathbf{X}}, t) = \hat{\hat{H}}_1 + \hat{\hat{H}}_2 + \hat{\hat{H}}_3 + \Gamma(t) = \Gamma(t) + \sum_{i=1}^{3}\hat{\hat{H}}, \tag{30}$$

where $\hat{\hat{H}}$, is the partial Hamiltonian of a quadratic system in x_i direction. And from (11) and (27) this partial Hamiltonian takes the form

$$\hat{\hat{H}}_i = \frac{1}{2m\rho_i^2}\hat{\hat{P}}_i^2 + \frac{m}{2\rho_i^2}\hat{X}_i\Delta_{ii}\hat{X}_i.$$

In order to delete the matrix $\mathbf{D}_1$ from the expression (27) of the Hamiltonian, and to have a simpler form similar to that of 3D free particle, we apply the following partial temporal transformation to each of the three partial Hamiltonians:

$$\frac{d\tau_i}{dt} = \frac{1}{\rho_i^2} \tag{31}$$

and write each Hamiltonian as

$$\hat{H}_i(\hat{\hat{P}}_i, \hat{X}_i, \tau_i) = \rho_i^2\hat{\hat{H}}_i(\hat{\hat{P}}_i, \hat{X}_i, t), \tag{32}$$

to obtain finally the Hamiltonian as

$$\hat{H} = \hat{H}_0 + \Gamma(t)$$

with

$$\hat{H}_0 = \sum_{i=1}^{3} \hat{H}_i = \frac{1}{2m}\hat{\tilde{\mathbf{P}}}^2 + \frac{m}{2}\hat{\mathbf{X}}\boldsymbol{\Delta}\hat{\mathbf{X}}. \tag{33}$$

The variation of the action operator is expressed then in the new representation as

$$\delta \hat{W} = \sum_{i=1}^{3}(\hat{\tilde{P}}_i''\delta\hat{X}_i'' - \hat{H}_i''\delta\tau_i'' - \hat{\tilde{P}}_i'\delta\hat{X}_i' + \hat{H}_i'\delta\tau_i') + \delta[\hat{F}_1'' - \hat{F}_1' + \hat{\tilde{F}}_1'' - \hat{\tilde{F}}_1' + \Lambda(\tau)] \tag{34}$$

with: $\Lambda = \int_{t'}^{t''} \Gamma dt$, and Γ verifies expression (20). If we consider $\boldsymbol{\Lambda} = 0$ in (29) it will be required to solve the following second order differential equation:

$$\ddot{\mathbf{T}} + \mathbf{T}\boldsymbol{\Omega}(t) = 0, \tag{35}$$

where

$$\boldsymbol{\Omega}(t) = 4\mathbf{D}^2\mathbf{G}_2 - 4\mathbf{G}_{31}^2 + 4\mathbf{D}^{-1}\dot{\mathbf{D}}\mathbf{G}_{31} - 2\mathbf{D}^{-2}\dot{\mathbf{D}}^2 + \mathbf{D}^{-1}\ddot{\mathbf{D}} - 2\dot{\mathbf{G}}_{31}. \tag{36}$$

The solution of this differential equation is related to the nature of the studied system as it depends on the expressions of the matrices $\mathbf{G}_i$. The system is finally transformed to a combination of three free particles, with one degree of freedom for each of them, and moving independently from each others in the three directions:

$$\hat{H} = \sum_{i=1}^{3} \hat{H}_i + \Gamma(t)$$

with

$$\hat{H}_i = \frac{\hat{\tilde{P}}_i^2}{2m}. \tag{37}$$

Consequently, we can write the variation of the action operator as

$$\delta \hat{W} = \sum_{i=1}^{3} \delta \hat{W}_i + \delta[\hat{F}_1'' - \hat{F}_1' + \hat{\tilde{F}}_i'' - \hat{\tilde{F}}_i' + \Lambda(t)] \tag{38}$$

with $\delta \hat{W}_i$ the variation of the action operator for a free particle moving in the direction x_i. Using Schwinger action principle after decoupling the system into three independent movements, we obtain the propagator as a multiplication of three propagators for free particles with an additional factor

$$\langle \mathbf{X}'', t'' | \mathbf{X}', t' \rangle = \prod_{i=1}^{3} (\langle X_i'', \tau_i'' | X_i', \tau_i' \rangle) \exp \frac{i}{\hbar}[F_1'' - F_1' + \tilde{F}_1'' - \tilde{F}_1' - \Lambda]. \tag{39}$$

Knowing that the propagator for a free particle moving in a direction x_i takes the form

$$\langle X_i'', \tau'' | X_i', \tau' \rangle = \left(\frac{2\pi i\hbar(\tau'' - \tau')}{m}\right)^{-1/2} \exp \frac{im(X_i'' - X_i')^2}{2\hbar(\tau'' - \tau')} \tag{40}$$

and from the partial temporal transformation (31) which leads to

$$\tau_i'' - \tau_i' = \int_{t'}^{t''} \frac{dt}{\rho_i^2}, \tag{41}$$

the canonical transformations (10) make it possible to write the propagator of the system as

$$\begin{aligned}
&\langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle \\
&= A \left(\frac{2\pi i \hbar}{m} \right)^{-3/2} \exp\left(\mathbf{f}_2'' \mathbf{x}'' - \mathbf{f}_2' \mathbf{x}' - \frac{i}{\hbar} \Lambda \right) \prod_{i=1}^{3} \left(\int_{t'}^{t''} \frac{dt}{\rho_i^2} \right)^{-1/2} \\
&\quad \times \exp\left(\frac{i}{4\hbar} \sum_{i=1}^{3} \left(\left(\sum_{j=1}^{3} \frac{R_{ij}''^{-1}(x_j'' - f_{1_j}'')}{\rho_i'' g_i''^{1/2}} - \frac{R_{ij}'^{-1}(x_j' - f_{1_j}')}{\rho_i' g_i'^{1/2}} \right)^2 \Big/ \int_{t'}^{t''} \frac{dt}{\rho_i^2} \right.\right. \\
&\quad \left.\left. - \frac{1}{g_i''} \left(2k_i'' - \frac{\dot{\rho}_i''}{\rho_i''} - \frac{\dot{g}_i''}{2g_i''} \right) (x_i'' - f_{1_i}'')^2 + \frac{1}{g_i'} \left(2k_i' - \frac{\dot{\rho}_i'}{\rho_i'} - \frac{\dot{g}_i'}{2g_i'} \right) (x_i' - f_{1_i}')^2 \right) \right). \qquad (42)
\end{aligned}$$

We use the following expression:

$$\lim_{\varepsilon \to 0} (\varepsilon \pi)^{-1/2} \exp\left(-\frac{(x'' - x')^2}{\varepsilon} \right) = \delta(x'' - x') \tag{43}$$

with the normalization condition to calculate the coefficient A [14]. We get finally the exact propagator, which represents the kernel of the general 3D TDQS with linear terms, in its final shape:

$$\begin{aligned}
&\langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle \\
&= (4\pi i \hbar)^{-3/2} \exp\left(\mathbf{f}_2'' \mathbf{x}'' - \mathbf{f}_2' \mathbf{x}' - \frac{i}{\hbar} \Lambda \right) \prod_{i=1}^{3} \left(\rho_i' \rho_i'' g_i'^{1/2} g_i''^{1/2} \int_{t'}^{t''} \frac{dt}{\rho_i^2} \right)^{-1/2} \\
&\quad \times \exp\left(\frac{i}{4\hbar} \sum_{i=1}^{3} \left(\left(\sum_{j=1}^{3} \frac{R_{ij}''^{-1}(x_j'' - f_{1_j}'')}{\rho_i'' g_i''^{1/2}} - \frac{R_{ij}'^{-1}(x_j' - f_{1_j}')}{\rho_i' g_i'^{1/2}} \right)^2 \Big/ \int_{t'}^{t''} \frac{dt}{\rho_i^2} \right.\right. \\
&\quad \left.\left. - \frac{1}{g_i''} \left(2k_i'' - \frac{\dot{\rho}_i''}{\rho_i''} - \frac{\dot{g}_i''}{2g_i''} \right) (x_i'' - f_{1_i}'')^2 + \frac{1}{g_i'} \left(2k_i' - \frac{\dot{\rho}_i'}{\rho_i'} - \frac{\dot{g}_i'}{2g_i'} \right) (x_i' - f_{1_i}')^2 \right) \right), \qquad (44)
\end{aligned}$$

where $\Lambda = \int_{t'}^{t''} \Gamma dt$, Γ verifies (20), and the vectors $\mathbf{f}_i$ and the values ρ_i are respectively the solutions of the differential equations (21) and (35).

The general solutions of the differential equations (21), (23), and (35), on which depends the propagator (44), are combinations of independent solutions depending on the arbitrary constants of integration. In the applications that will be illustrated in the following sections, it could be verified the existence of an invariance property in the system propagator called "gauge invariance", related to the choice of these constants.

As applications of the general result obtained in (44), we are going to treat, in the following sections, the three general cases explained above: a system with only one varying coupling and a partial isotropy in the quadratic terms, a system with 3 varying couplings and isotropic quadratic terms, and an anisotropic quadratic system without any coupling.

The vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ in the canonical transformations (10) are used in order to delete the linear terms in the Hamiltonian (8) of the general 3D system. If we consider the following particular case by taking, $\mathbf{z}_1 = 0$, $\mathbf{z}_2 = 0$ and $\delta(t) = 0$ in the Hamiltonian (8) such as

$$\hat{H} = \hat{\mathbf{p}}\mathbf{G}_1\hat{\mathbf{p}} + \hat{\mathbf{x}}\mathbf{G}_2\hat{\mathbf{x}} + \hat{\mathbf{x}}\mathbf{G}_3\hat{\mathbf{p}} + \hat{\mathbf{p}}\mathbf{G}_4\hat{\mathbf{x}}, \tag{45}$$

the canonical transformation (10) becomes after dropping $\mathbf{f}_1$ and $\mathbf{f}_2$:

$$\begin{cases} \hat{\mathbf{x}} = (2m)^{1/2}\mathbf{DRT}\hat{\mathbf{X}}, \\ \hat{\mathbf{p}} = (2m)^{-1/2}\mathbf{D}^{-1}\mathbf{RT}^{-1}\hat{\mathbf{P}}. \end{cases} \tag{46}$$

We will not do again all the steps of the previous paragraph, we will only use directly the result (44) obtained after solving differential equations (23) and (35), and we will make the substitutions considered above. The propagator of the system takes then the following expression:

$$\begin{aligned} \langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle &= (4\pi i \hbar)^{-3/2} \prod_{i=1}^{3} \left(\rho_i' \rho_i'' g_i''^{1/2} g_i''^{1/2} \int_{t'}^{t''} \frac{dt}{\rho_i^2} \right)^{-1/2} \\ &\quad \times \exp\left(\frac{i}{4\hbar} \sum_{i=1}^{3} \left(\left(\sum_{j=1}^{3} \frac{R_{ij}''^{-1} x_j''}{\rho_i'' g_i''^{1/2}} - \frac{R_{ij}'^{-1} x_j'}{\rho_i' g_i'^{1/2}} \right)^2 \Big/ \int_{t'}^{t''} \frac{dt}{\rho_i^2} \right.\right. \\ &\quad \left.\left. - \frac{1}{g_i''} \left(2k_i'' - \frac{\dot{\rho}_i''}{\rho_i''} - \frac{\dot{g}_i''}{2g_i''} \right) x_i''^2 + \frac{1}{g_i'} \left(2k_i' - \frac{\dot{\rho}_i'}{\rho_i'} - \frac{\dot{g}_i'}{2g_i'} \right) x_i'^2 \right) \right), \end{aligned} \tag{47}$$

where ρ_i is the solution of the differential equation (35).

3 The Propagator for 3D TDQS with One Varying Coupling

3.1 General Case

It is imposed to the coupling in the system to be in the same directions as the partial isotropy, in order to apply expression (44) here immediately. For this reason, we introduce in this section the coupling and the partial isotropy in quadratic terms in x_1 and x_2 directions, whereas the anisotropy is preserved for the linear terms in each of the three directions. In this case, the matrices $\mathbf{G}_i$, in the Hamiltonian (8), and expressed in (9), take the following forms:

$$\begin{aligned} &\mathbf{G}_1 = \begin{pmatrix} g(t) & 0 & 0 \\ 0 & g(t) & 0 \\ 0 & 0 & g_3(t) \end{pmatrix}, \qquad \mathbf{G}_2 = \begin{pmatrix} h(t) & 0 & 0 \\ 0 & h(t) & 0 \\ 0 & 0 & h_3(t) \end{pmatrix}, \qquad \mathbf{G}_3 = \mathbf{G}_4^T, \\ &\mathbf{G}_3 = \mathbf{G}_{31} + \mathbf{G}_{32}, \qquad \mathbf{G}_{31} = \begin{pmatrix} k(t) & 0 & 0 \\ 0 & k(t) & 0 \\ 0 & 0 & k_3(t) \end{pmatrix}, \qquad \mathbf{G}_{32} = \frac{1}{2}\begin{pmatrix} 0 & l(t) & 0 \\ -l(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{48}$$

We will apply to this system the same canonical transformations as that of (10), where the time dependent matrices **D** and **T** verify the followings formulas:

$$\mathbf{T}=\begin{pmatrix}\rho(t) & 0 & 0\\ 0 & \rho(t) & 0\\ 0 & 0 & \rho(t)\end{pmatrix}\quad\text{and}\quad\mathbf{D}=\begin{pmatrix}g^{1/2}(t) & 0 & 0\\ 0 & g^{1/2}(t) & 0\\ 0 & 0 & g_3^{1/2}(t)\end{pmatrix},\quad\text{where } \mathbf{D}^2=\mathbf{G}_1 \tag{49}$$

and as the angular momentum, which involves the coupling terms, is in the x_3 direction, the rotation in the canonical transformations will be chosen in the same direction, in order to eliminate this coupling, so that the matrix **R** takes the form:

$$\mathbf{R}=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{50}$$

The differential equation (23) is then reduced to the solution of the differential equations (23) is then

$$\dot{\theta}=-l(t). \tag{51}$$

We make these substitutions in (44) to get finally the propagator for a general 3D TDQS, with a one varying hybrid coupling and linear terms, as

$$\begin{aligned}
&\langle\mathbf{x}'',t''|\mathbf{x}',t'\rangle\\
&\quad=(4\pi i\hbar)^{-3/2}\exp\left(f''_{2_1}x''_1-f'_{2_1}x'_1+f''_{2_2}x''_2-f'_{2_2}x'_2-\frac{i}{\hbar}\Lambda\right)\\
&\qquad\times\left(g''^{1/2}g'^{1/2}\rho''\rho'\int_{t'}^{t''}\frac{dt}{\rho^2}\right)^{-1}\exp\left(\frac{i}{4\hbar}\left(\left(\left(\frac{(x''_1-f''_{1_1})\cos\theta''-(x''_2-f''_{1_2})\sin\theta''}{\rho''g''^{1/2}}\right.\right.\right.\right.\\
&\qquad-\left.\frac{(x'_1-f'_{1_1})\cos\theta'-(x'_2-f'_{1_2})\sin\theta'}{\rho'g'^{1/2}}\right)^2+\left(\frac{(x''_1-f''_{1_1})\sin\theta''+(x''_2-f''_{1_2})\cos\theta''}{\rho''g''^{1/2}}\right.\\
&\qquad-\left.\left.\frac{(x'_1-f'_{1_1})\sin\theta'+(x'_2-f'_{1_2})\cos\theta'}{\rho'g'^{1/2}}\right)^2\right)\Bigg/\int_{t'}^{t''}\frac{dt}{\rho^2}\\
&\qquad-\frac{1}{g''}\left(2k''-\frac{\dot{\rho}''}{\rho''}-\frac{\dot{g}''}{2g''}\right)((x''_1-f''_{1_1})^2+(x''_2-f''_{1_2})^2)\\
&\qquad+\left.\left.\frac{1}{g'}\left(2k'-\frac{\dot{\rho}'}{\rho'}-\frac{\dot{g}'}{2g'}\right)((x'_1-f'_{1_1})^2+(x'_2-f'_{1_2})^2)\right)\right)\\
&\qquad\times\left(g_3''^{1/2}g_3'^{1/2}\rho''_3\rho'_3\int_{t'}^{t''}\frac{dt}{\rho_3^2}\right)^{-1/2}\exp\left(\frac{i}{4\hbar}\left(\left(\frac{(x''_3-f''_{1_3})}{\rho''_3g_3''^{1/2}}-\frac{(x'_3-f'_{1_3})}{\rho'_3g_3'^{1/2}}\right)^2\right.\right.\\
&\qquad\Bigg/\int_{t'}^{t''}\frac{dt}{\rho_3^2}-\frac{1}{g''_3}\left(2k''_3-\frac{\dot{\rho}''_3}{\rho''_3}-\frac{\dot{g}''_3}{2g''_3}\right)(x''_3-f''_{1_3})^2\\
&\qquad+\left.\left.\frac{1}{g'_3}\left(2k'_3-\frac{\dot{\rho}'_3}{\rho'_3}-\frac{\dot{g}'_3}{2g'_3}\right)(x'_3-f'_{1_3})^2+f''_{2_3}x''_3-f'_{2_3}x'_3\right)\right),
\end{aligned}\tag{52}$$

where $\Lambda=\int_{t'}^{t''}\Gamma dt$, and Γ verifies (20) and the vectors $\mathbf{f}_i$, and the value ρ_i, are, respectively, the solutions of the differential equations (21), and (35).

As the canonical transformation does not mix the first two degrees of freedom with the third one, the propagator is in fact the multiplication of two propagators for two independent systems: a two dimensional quadratic system with a varying hybrid coupling and a certain isotropy in the directions x_1 and x_2, and a general one dimensional quadratic system in the direction x_3. It can be seen that the propagator of this later takes exactly the form obtained with this technique in our first generalization for 1D systems [14]. In this regard, we have to deal only with a two dimensional problem.

Particularly, and as in the previous section, we will consider $\mathbf{z}_1 = 0$, $\mathbf{z}_2 = 0$, and $\delta(t) = 0$ in the Hamiltonian (8), what leads to the quadratic Hamiltonian (45) without linear terms. Thus, we are going to drop from the canonical transformations (10) the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ used just to eliminate the linear terms in the Hamiltonian (8). In this case, the propagator (52) turns to be equal to

$$\begin{aligned}
\langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle &= (4\pi i\hbar)^{-3/2} \left(g''^{1/2} g'^{1/2} \rho'' \rho' \int_{t'}^{t''} \frac{dt}{\rho^2} \right)^{-1} \left(g_3''^{1/2} g_3'^{1/2} \rho_3'' \rho_3' \int_{t'}^{t''} \frac{dt}{\rho_3^2} \right)^{-1/2} \\
&\quad \times \exp\Bigg(\frac{i}{4\hbar} \Bigg(\left(\frac{x_3''}{\rho_3'' g_3''^{1/2}} - \frac{x_3'}{\rho_3' g_3'^{1/2}} \right)^2 \Bigg/ \int_{t'}^{t''} \frac{dt}{\rho_3^2} \\
&\quad - \frac{1}{g_3''} \left(2k_3'' - \frac{\dot{\rho}_3''}{\rho_3''} - \frac{\dot{g}_3''}{2g_3''} \right) x_3''^2 + \frac{1}{g_3'} \left(2k_3' - \frac{\dot{\rho}_3'}{\rho_3'} - \frac{\dot{g}_3'}{2g_3'} \right) x_3'^2 \Bigg) \Bigg) \\
&\quad \times \exp\Bigg(\frac{i}{4\hbar} \Bigg(\Bigg(\left(\frac{x_1'' \cos\theta'' - x_2'' \sin\theta''}{\rho'' g''^{1/2}} - \frac{x_1' \cos\theta' - x_2' \sin\theta'}{\rho' g'^{1/2}} \right)^2 \\
&\quad + \left(\frac{x_1'' \sin\theta'' + x_2'' \cos\theta''}{\rho'' g''^{1/2}} - \frac{x_1' \sin\theta' + x_2' \cos\theta'}{\rho' g'^{1/2}} \right)^2 \Bigg) \\
&\quad \Bigg/ \int_{t'}^{t''} \frac{dt}{\rho^2} - \frac{1}{g''} \left(2k'' - \frac{\dot{\rho}''}{\rho''} - \frac{\dot{g}''}{2g''} \right) (x_1''^2 + x_2''^2) \\
&\quad + \frac{1}{g'} \left(2k' - \frac{\dot{\rho}'}{\rho'} - \frac{\dot{g}'}{2g'} \right) (x_1'^2 + x_2'^2) \Bigg) \Bigg),
\end{aligned} \tag{53}$$

where ρ_i are the solutions of the differential equation (35). As previously in the same section, the same remark can be done here for the two independent propagators that form this propagator.

3.2 Applications

3.2.1 The Propagator for a Charged Particle in a Constant External Magnetic Field

We will consider this well known case of a particle of charge e and mass m, submitted to a constant magnetic field B, in the direction x_3, in order to check the exactness of our general result and confront it to results obtained in other studies. The Hamiltonian of the system is

$$\begin{aligned}
\hat{H} &= \frac{1}{2m} \left(\left(\hat{p}_1 + \frac{eB}{2c} \hat{x}_2 \right)^2 + \left(\hat{p}_2 + \frac{eB}{2c} \hat{x}_1 \right)^2 + \hat{p}_3^2 \right) \\
&= \frac{1}{2m} (\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2) + \frac{m}{2} \left(\frac{eB}{2mc} \right)^2 (\hat{x}_1^2 + \hat{x}_2^2) - \frac{eB}{2mc} (\hat{p}_2 \hat{x}_1 - \hat{p}_1 \hat{x}_2).
\end{aligned} \tag{54}$$

This Hamiltonian is quadratic with a hybrid coupling and an isotropy in x_1 and x_2 directions and without linear terms. Subsequently, it is useful to apply immediately, in this case, the result (53). From expressions (45) and (48) we can make these considerations:

$$l = -\frac{eB}{2mc} = -\frac{\omega_c}{2}, \qquad g_i = \frac{1}{2m}, \qquad h = \frac{m}{2}\left(\frac{\omega_c}{2}\right), \qquad h_3 = k_i = 0. \tag{55}$$

To determine the quantities θ and ρ_i in the propagator (53), we have to find the appropriate canonical transformations using the differential equations (51) and (35), which turn into the following formulas after substituting the expressions (55):

$$\dot{\theta} = -l = \frac{\omega_c}{2} \tag{56}$$

and

$$\begin{cases} \ddot{\rho} + \left(\frac{\omega_c}{2}\right)^2 \rho = 0, \\ \ddot{\rho}_3 = 0. \end{cases} \tag{57}$$

The general solutions for these equations are

$$\theta = \frac{\omega_c}{2} t + \theta_c \tag{58}$$

and

$$\begin{cases} \rho = C_1 \cos\left(\frac{\omega_c}{2} t\right) + C_2 \sin\left(\frac{\omega_c}{2} t\right), \\ \rho_3 = at + b, \end{cases} \tag{59}$$

θ_c, C_1, C_2, a and b are arbitrary constants of integration. As there exists a "gauge invariance" in the propagator related to the choice of these constants, it would be better to use the simplest solutions not to complicate the calculations, it means to choose these constants so that: $\rho = \cos(\omega_c t/2)$, $\rho_3 = 1$, and $\theta = \omega_c t/2$. We substitute (58) and (59) in the propagator (53) to obtain finally the propagator of a charged particle in a constant magnetic field as

$$\begin{aligned} \langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle = {} & \left(\frac{2\pi i\hbar T}{m}\right)^{-3/2} \left(\frac{\omega_c T}{2\sin(\omega_c T/2)}\right) \\ & \times \exp\left(\frac{im}{2\hbar}\left(\frac{\omega_c}{2}(\cot(\omega_c T/2)((x_1'' - x_1')^2 + (x_2'' - x_2')^2)\right.\right. \\ & \left.\left. + 2(x_2'' x_1' - x_1'' x_2')) + \left(\frac{x_3'' - x_3'}{T}\right)^2\right)\right), \end{aligned} \tag{60}$$

where $T = t'' - t'$. And this is exactly the same result obtained with path integrals [21] and it is a certification of our technique exactness. The remark that could be made here is that it was not necessary to substitute ρ_3 in the propagator (53) and that we could immediately, and from the beginning, consider the propagator of the system as a multiplication of a propagator for a free particle moving in x_3 direction, and that of a 2D system in x_1 and x_2 directions.

3.2.2 The Propagator for a Charged Particle in a Constant External Magnetic Field and a Scalar Constant Potential

In the same way as in the previous paragraph, we will consider the case of a particle of charge e and mass m, submitted to a constant magnetic field B in the direction x_3, and to a scalar potential so that the Hamiltonian of the system takes the form:

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\left(\left(\hat{p}_1 + \frac{eB}{2c}\hat{x}_2\right)^2 + \left(\hat{p}_2 + \frac{eB}{2c}\hat{x}_1\right)^2 + \hat{p}_3^2\right) + \frac{m}{2}\Omega^2(\hat{x}_1^2 + \hat{x}_2^2) \\ &= \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2) + \frac{m}{2}\left(\left(\frac{eB}{2mc}\right)^2 + \Omega^2\right)(\hat{x}_1^2 + \hat{x}_2^2) - \frac{eB}{2mc}(\hat{p}_2\hat{x}_1 - \hat{p}_1\hat{x}_2).\end{aligned} \tag{61}$$

The unique difference with the previous case is that

$$h = \frac{m}{2}\left(\left(\frac{\omega_c}{2}\right)^2 + \Omega^2\right) = \frac{m}{2}\overline{\omega}^2, \tag{62}$$

which introduces a change in the solution of the differential equation (35), so that

$$\ddot{\rho} + \varpi^2\rho = 0. \tag{63}$$

Consequently, the solution is expressed as

$$\rho = C_1 \cos\overline{\omega}t + C_2 \sin\overline{\omega}t. \tag{64}$$

Hence the propagator (60) for a charged particle submitted to a constant magnetic field without a scalar potential becomes here with the constant scalar potential as follows:

$$\begin{aligned}\langle \mathbf{x}'', t''|\mathbf{x}', t'\rangle = {} & \left(\frac{2\pi i\hbar T}{m}\right)^{-3/2}\left(\frac{\varpi T}{\sin(\varpi T)}\right) \\ & \times \exp\left(\frac{im}{2\hbar}\left(\frac{\varpi}{\sin(\varpi T)}(\cos(\varpi T)(x_1''^2 + x_1'^2 + x_2''^2 + x_2'^2)\right.\right. \\ & - 2\cos(\omega_c T/2)(x_1''x_1' + x_2''x_2') \\ & \left.\left. + 2\sin(\omega_c T/2)(x_2''x_1' - x_1''x_2')) + \left(\frac{x_3'' - x_3'}{T}\right)^2\right)\right),\end{aligned} \tag{65}$$

where $T = t'' - t'$. An other study applying Schwinger action principle in other way by resolving indirectly Heisenberg equations of motions got the same result [12]. The remark we can make here is that the result above is available in the case where the magnetic field is variable. Its dependence on the time appears in ω_c from (56).

From the propagator (52) we can find directly a diversity of propagators for 3D TDQS submitted to variable magnetic fields, and variable scalar potentials, and driven by variable external forces, like: harmonic oscillator with time-dependent frequency, harmonic oscillator with time-dependent mass and frequency, damped harmonic oscillator, Calidora-Kanai oscillator, etc.

4 Propagator for 3D Anisotropic TDQS

For this system, and this condition of 3D anisotropy, each of the matrices $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{G}_3$, related to the quadratic terms, are anisotropic ones. Subsequently, they preserve there forms expressed in (9). The condition that should be verified here, in order to be able to use directly result (44), is that there would not be any coupling term, i.e.

$$\mathbf{G}_{32} = 0. \tag{66}$$

This means also that there would not be a rotation in the canonical transformations (10). Rotation matrix is then equal to identity matrix. With all these considerations, $\mathbf{D}$ and $\mathbf{T}$ in these transformations are also anisotropic and verify expression (11). Subsequently, the propagator (44) becomes

$$\begin{aligned}
&\langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle \\
&\quad = (4\pi i\hbar)^{-3/2} \exp \frac{i}{\hbar}\left(\mathbf{f}''_2 \mathbf{x}'' - \mathbf{f}'_2 \mathbf{x}' - \frac{i}{\hbar}\Lambda\right) \times \prod_{i=1}^{3}\left(\rho'_i \rho''_i g_i'^{1/2} g_i''^{1/2} \int_{t'}^{t''} \frac{dt}{\rho_i^2}\right)^{-1/2} \\
&\qquad \times \exp\left(\frac{i}{4\hbar}\sum_{i=1}^{3}\left(\left(\frac{(x''_i - f''_{1_i})}{\rho''_i g_i''^{1/2}} - \frac{(x'_i - f'_{1_i})}{\rho'_i g_i'^{1/2}}\right)^2 \Big/ \int_{t'}^{t''} \frac{dt}{\rho_i^2}\right.\right. \\
&\qquad \left.\left. - \frac{1}{g''_i}\left(2k''_i - \frac{\dot{\rho}''_i}{\rho''_i} - \frac{\dot{g}''_i}{2g''_i}\right)(x''_i - f''_{1_i})^2 + \frac{1}{g'_i}\left(2k'_i - \frac{\dot{\rho}'_i}{\rho'_i} - \frac{\dot{g}'_i}{2g'_i}\right)(x'_i - f'_{1_i})^2\right)\right).
\end{aligned} \tag{67}$$

This is the propagator for anisotropic 3D TDQS with varying linear terms without any coupling. And it is clear that this propagator is formed from three independent propagators for general linear quadratic systems, and each of them takes exactly the expression obtained in our first generalisation for 1D TDQS using the same technique [14]. And this is an other verification for the efficiency of the generalization of Schwinger action principle via canonical transformations, which is the aim of all this work.

This result is useful to determine immediately propagators for 3D TDQS with independent movements in each of the three directions.

5 Propagators for 3D TDQS with Three Varying Anisotropic Hybrid Couplings

In this section, we are going to evaluate the propagator for a system with three different varying couplings, what was not dealt with, using Schwinger method, before that. With the canonical transformations we have introduced till now, only 3D isotropy in the quadratic terms of the Hamiltonian of the system gives us the possibility to introduce three different couplings. Our technique, using Schwinger action principle by means of canonical transformations, makes it possible to determine immediately and easily the propagator of the system, despite of the complexity involved by the coupling terms. With this system, the matrices $\mathbf{G}_i$ in the Hamiltonian (8) should be expressed as

$$\begin{aligned}
&\mathbf{G}_1 = g(t)\mathbf{I}, \qquad \mathbf{G}_2 = h(t)\mathbf{I}, \qquad \mathbf{G}_3 = \mathbf{G}_4^T, \\
&\mathbf{G}_3 = \mathbf{G}_{31} + \mathbf{G}_{32}, \qquad \mathbf{G}_{31} = k(t)I, \qquad \mathbf{G}_{32} = \frac{1}{2}\begin{pmatrix} 0 & l_3(t) & -l_2(t) \\ -l_3(t) & 0 & l_(t) \\ l_2(t) & -l_1(t) & 0 \end{pmatrix},
\end{aligned} \tag{68}$$

I here is the identity matrix. This involves in canonical transformations (10):

$$\mathbf{T} = \rho(t)\mathbf{I} \quad \text{and} \quad \mathbf{D} = g^{1/2}\mathbf{I}, \text{ where } \mathbf{D}^2 = \mathbf{G}_1. \tag{69}$$

The coupling terms are expressed in the matrix $\mathbf{G}_{32}$ which contains three different couplings. And as the rotation in the canonical transformations should be made in the same directions as for the couplings in order to delete them, the rotation matrix in the canonical transformations is expressed as

$$\begin{aligned} &R = \prod_{l=1}^{3} R_l = (R^T)^{-1}, \quad \text{where} \\ &R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}, \\ &R_3 = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & -\cos\theta_3 & \cos\theta_3 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{70}$$

Therefore, solving differential equation (23) requires the solution of the following system of differential equations:

$$\begin{cases} \dot\theta_1 = -l_1 - (l_2 \sin\theta_1 + l_3\cos\theta_1)\mathrm{tg}\,\theta_2, \\ \dot\theta_2 = -l_2\cos\theta_1 + l_3\sin\theta_1, \\ \dot\theta_3 = -l_2 \dfrac{\sin\theta_1}{\cos\theta_2} - l_3 \dfrac{\cos\theta_1}{\cos\theta_2}. \end{cases} \tag{71}$$

And differential equations (35) and (36) are reduced in this case to

$$\ddot\rho = \Omega(t)\rho = 0, \tag{72}$$

with

$$\Omega(t) = 4g(t)h(t) - 4k^3(t) + 2\frac{\dot g(t)k(t)}{g(t)} + \frac{\ddot g}{2g(t)} - 2\dot k(t).$$

Finally, from the general propagator (44), the exact propagator for the isotropic 3D TDQS, with three anisotropic varying couplings and linear terms, is written as

$$\begin{aligned} \langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle = {} & \left(4\pi i\hbar g'^{1/2} g''^{1/2} \rho' \rho'' \int_{t'}^{t''} \frac{dt}{\rho^2} \right)^{-3/2} \\ & \times \exp\Bigg(\frac{i}{\hbar} \Bigg(\left(\frac{\mathbf{R}''^{-1}(\mathbf{x}'' - \mathbf{f}_1'')}{2\rho'' g''^{1/2}} - \frac{\mathbf{R}'^{-1}(\mathbf{x}' - \mathbf{f}_1')}{2\rho' g'^{1/2}} \right)^2 \Big/ \int_{t'}^{t''} \frac{dt}{\rho^2} \\ & - \frac{1}{4g''}\left(2k'' - \frac{\dot\rho''}{\rho''} - \frac{\dot g''}{2g''} \right)(\mathbf{x}'' - \mathbf{f}_1'')^2 \\ & + \frac{1}{4g'}\left(2k' - \frac{\dot\rho'}{\rho'} - \frac{\dot g'}{2g'} \right)(\mathbf{x}' - \mathbf{f}_1')^2 + \mathbf{f}_2''\mathbf{x}'' - \mathbf{f}_2'\mathbf{x}' - \Lambda \Bigg) \Bigg), \end{aligned} \tag{73}$$

where $\mathbf{f}_i$, $\mathbf{R}$, and ρ, are obtained directly in resolving, respectively, the differential equations (21), (71), and (72). This result is of great importance as these systems with deferent varying couplings were not dealt with, in other researches, using Schwinger method. Despite of the complexity caused by the varying anisotropy in coupling terms and linear terms, we have obtained an immediate result, and the applications would be made in resolving the differential equations (21), (71), and (72) with the less possible difficulty comparing to other techniques.

Now we are going to conclude the propagator for the quadratic Hamiltonian without linear terms which is expressed in (45). After using in expression (47) the condition of isotropy related to this kind of systems, as explained above in the previous paragraph, we get the propagator of the system as

$$\begin{aligned}\langle \mathbf{x}'', t''|\mathbf{x}', t'\rangle &= \left(4\pi i\hbar g'^{1/2} g''^{1/2}\rho'\rho'' \int_{t'}^{t''} \frac{dt}{\rho^2}\right)^{-3/2} \\ &\quad\times \exp\left(\frac{i}{\hbar}\left(\left(\frac{\mathbf{R}''^{-1}\mathbf{x}''}{2\rho'' g''^{1/2}} - \frac{\mathbf{R}'^{-1}\mathbf{x}'}{2\rho' g'^{1/2}}\right)^2 \Big/ \int_{t'}^{t''} \frac{dt}{\rho^2}\right.\right. \\ &\quad\left.\left. - \frac{1}{4g''}\left(2k'' - \frac{\dot\rho''}{\rho''} - \frac{\dot g''}{2g''}\right)\mathbf{x}''^2 + \frac{1}{4g'}\left(2k' - \frac{\dot\rho'}{\rho'} - \frac{\dot g'}{2g'}\right)\mathbf{x}'^2\right)\right). \quad (74)\end{aligned}$$

We can remark here also that the difference between this result and that of the previous paragraph is the omission of $\mathbf{f}_1$ and $\mathbf{f}_2$ which are vectors used in the canonical transformations to delete linear terms.

6 Summary

In this work, we were devoted to develop a technique using Schwinger action principle by means of canonical transformations, in order to determine exact propagators for general 3D TDQS, with linear terms. The elimination of existing couplings by a canonical transformation containing a rotation matrix, with the diagonalization of the matrices related to the quadratic terms in the Hamiltonian, transforms the problem to that of 3D free particle, what enables us to avoid the resolution of Heisenberg equations of motion .

In this generalization, we obtained a general form of the propagator for 3D TDQS, related to some temporal factors, introduced by the canonical transformations, and determined in resolving certain differential equations.

The power of this technique resides in the possibility that it gives to obtain directly exact expressions of propagators for general 3D systems, containing multiple varying and anisotropic hybrid couplings, with the resolution of the simpler possible differential equations.

With the results we have reached till now, to be able to determine the propagator for the general 3D TDQS, using our technique, the addition of a coupling term in certain directions imposes an isotropy in the quadratic terms for the same directions, otherwise dynamical or static couplings immerge with the canonical transformations. Once verifying this condition, whatever is the anisotropy in the linear terms, the problem remains a solvable one. For this reason, 3 kinds of systems were treated here:

(1) A system with one varying hybrid coupling and partially isotropic quadratic terms in the same directions. The efficiency of this method and of the general solution was checked here, in confronting with other works the result obtained in two well known particular cases: a charged particle submitted to a constant magnetic field, and a charged particle submitted to constant vector potential and scalar potential.

(2) An anisotropic system with no coupling terms. The propagator of this system is formed of three independent propagators, each of them is the propagator of a 1D TDQS in a certain direction, and it agrees perfectly with the result we have obtained in a previous work for this kind of systems [14].
(3) A system with three varying anisotropic hybrid couplings with an isotropy in all of the quadratic terms. This isotropy is not imposed to the linear terms.

In this paper, we have added successfully an other stone to the generalization hopped for TDQS by means of Schwinger action principle, and using canonical transformations. This step is promising, and it paves the way to an extension of this technique to N-dimensional systems for more complicated situations, and more general systems with varying anisotropic dynamical and static couplings, in adding other terms to the canonical transformations applied to the studied systems. Therefore, N-dimensional systems, and Hamiltonians with coulombian potentials or singular perturbations would be solvable with Schwinger action principle via canonical transformations.

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